

The translation operator for self-projective coalgebras

William Chin
DePaul University, Chicago Illinois 60614

Abstract

We describe the transpose operator for self-projective and symmetric coalgebras in terms of the syzygy and Nakayama functors.

1 Introduction and Preliminaries

The Auslander-Reiten (AR) translation plays a fundamental role in the representation theory of coalgebras and algebras. Given a right comodule M , under certain finiteness conditions, there exists an almost split sequence

$$0 \rightarrow M \rightarrow E \rightarrow \tau(M) \rightarrow 0$$

where $\tau(M)$ is called the translation of M . For many right semiperfect coalgebras, the almost split sequence exists for all non-projective finite-dimensional right comodules M [CKQ][CKQ2]. In these case the AR quiver exists, and the translation induces an important special type of automorphism of the stable quiver. An introduction to AR theory can be found the books [ARS][ASS].

A coalgebra is said to be right self-projective if it is projective as a right comodule over itself. A coalgebra is simply called self-projective if it is both left and right self-projective. Self-projective coalgebras are also known as quasi-co-Frobenius (qcF) coalgebras. A coalgebra C is said to be *right co-Frobenius* if it embeds to its \mathbb{k} -linear dual DC as a right DC -module. Facts and references concerning these types of coalgebras may be found in [DNR]. Basic information and references concerning the representation theory of coalgebras and path coalgebras of quivers can be found in [Ch].

A coalgebra C is said to be *symmetric* [CDN] if it is isomorphic to its \mathbb{k} -linear dual DC as a DC, DC -bimodule. A symmetric coalgebra is necessarily semiperfect, i.e. it indecomposable injectives are finite-dimensional. It turns out that if C is a symmetric coalgebra, then its basic coalgebra is also symmetric. In this article we focus on self-projective and symmetric coalgebras. In these cases, the translation operator takes a particularly nice form. We show that the translation given by $\Omega^{-2}\nu = \nu\Omega^{-2}$ for self-projective

coalgebras where ν is the Nakayama functor and Ω^{-2} is the second cosyzygy. In the symmetric case ν is naturally isomorphic to the identity functor on finite-dimensional comodules.

We present an example of a pointed self-projective serial coalgebra which is not symmetric and compute the translation, Nakayama functor, and syzygies. The coalgebra is a subcoalgebra of a quiver of type A_∞ . We then summarize some results from [Ch2] on representations of coordinate Hopf algebra $C = \mathbb{k}_\zeta[SL(2)]$ at a root of unity where \mathbb{k} is a field of characteristic zero. Here we show that C is a symmetric coalgebra. The results concerning the translation are illustrated in the almost split sequences and AR quiver.

A theory in a very different direction, but which also enables the computation of the translation and almost split sequences on the level of Grothendieck groups of comodules was recently developed in [CS]. There the theory of Coxeter transformations is developed, with a focus on hereditary coalgebras. In the case of path coalgebras with locally finite quivers, it is shown that the image of the translation of a finite-dimensional indecomposable comodule in the Grothendieck group is given by the Coxeter transformation, which is determined by a possibly infinite invertible Cartan matrix.

We shall let C denote a coalgebra over the fixed base field \mathbb{k} . A right C -comodule M is said to be *quasifinite* if $\text{Hom}^C(F, M)$ is finite-dimensional for all finite-dimensional comodules F . M is said to be *quasifinitely copresented* if it has an injective copresentation $0 \rightarrow N \rightarrow I_0 \rightarrow I_1$ with quasifinite injectives I_i . \mathcal{M}^C , \mathcal{M}_f^C , \mathcal{M}_q^C , \mathcal{M}_{qc}^C shall denote the category of right C -comodules, finite-dimensional right C -comodules, quasifinite right C -comodules and quasifinitely copresented right C -comodules, respectively. The \mathbb{k} -linear dual $\text{Hom}_{\mathbb{k}}(-, \mathbb{k}) : \mathcal{M}_f^C \rightarrow \mathcal{M}_f^{C^{op}}$ is denoted by $D : \mathcal{M}_f^C \rightarrow \mathcal{M}_f^{C^{op}}$. The cohom functor $h_C(-, -) : \mathcal{M}_q^C \rightarrow \mathcal{M}_{\mathbb{k}}$ is defined to be

$$h_C(M, N) = \lim_{\rightarrow} \text{DHom}(N_\lambda, M)$$

for $M, N \in \mathcal{M}^C$ where N_λ is the direct system of finite-dimensional comodules of N indexed by λ .

1.1 Symmetric and Self-projective coalgebras

Let N be a quasi-finite right C -comodule and let $I(N)$ denote its injective envelope. Recall [CKQ] that the functor $*$: $\mathcal{M}_q^C \rightarrow {}^C\mathcal{M}$ is defined by $h_C(-, C)$. The functor $*$ restricts to a duality on quasifinite injectives and is the coalgebraic version of the functor $\text{Hom}_R(-, R)$ for R -modules. The transpose operator Tr is defined in [CKQ] via the minimal injective copresentation

$$0 \rightarrow \text{Tr}N \rightarrow I_1^* \rightarrow I_0^*$$

where $0 \rightarrow N \rightarrow I_0 \rightarrow I_1$ is a minimal injective copresentation of N . In general, the translation operator is defined to be $D\text{Tr}(N)$ on comodules N

such that $\text{Tr}N$ is finite-dimensional. Here we are concerned with semiperfect coalgebras and will assume comodules are finite-dimensional so that the translation is always defined. The cosyzygy operator Ω^{-1} is defined as the cokernel in the exact sequence

$$0 \rightarrow N \rightarrow I(N) \rightarrow \Omega^{-1}(N) \rightarrow 0.$$

If C is right semiperfect, the syzygy operator Ω is defined by the exact sequence

$$0 \rightarrow \Omega(N) \rightarrow P(N) \rightarrow N \rightarrow 0$$

where $P(N)$ denotes the projective cover of N . For semiperfect coalgebras we note that these functors are inverse equivalences between the categories of finite-dimensional comodules modulo injectives and projectives respectively:

$$\overline{\mathcal{M}}_f^C \xrightleftharpoons[\Omega^{-1}]{\Omega} \underline{\mathcal{M}}_f^C.$$

We collect some facts about comodule functors. The reader is referred to [CDN] for a development of the theory of symmetric coalgebras, and the book [DNR] for semiperfect and self-projective (i.e. quasi-co-Frobenius) coalgebras. Here we just point out that symmetric \Rightarrow self-projective \Rightarrow semiperfect (one-sided and two-sided versions). In addition, it is recently be shown that one-sided self-projective implies semiperfect [Iov].

Lemma 1.1.1 *Let $C = \oplus_i P_i$ be a right self-projective coalgebra with projective-injective indecomposable right coideals P_i . Then for a finite-dimensional comodule N ,*

- (a) $N^* = \oplus_i D \text{Hom}(P_i, N)$
- (b) $*: \mathcal{M}_q^C \rightarrow {}^C \mathcal{M}$ is an exact functor.
- (c) $\nu: \mathcal{M}_f^C \rightarrow {}^C \mathcal{M}_f$ is an exact functor.

Proof. The first statement follows directly from the definition of the cohom functor since C is plainly the direct limit of the finite-dimensional projective summands P_i . The remaining assertions follows directly, since the P_i are projective and the \mathbb{k} -linear dual D is exact. ■

Proposition 1.1.2 (a) *For a left self-projective coalgebra C , the translation is naturally isomorphic to $\Omega^{-2}\nu$.*

(b) *For a right self-projective coalgebra C , the translation is naturally isomorphic to $\nu\Omega^{-2}$.*

(c) *For a symmetric coalgebra C , the translate is naturally isomorphic to Ω^{-2} .*

Proof. Let N be an indecomposable non-injective finite-dimensional right C -comodule. Then the transpose is defined by $0 \rightarrow \text{Tr}N \rightarrow I_1^* \rightarrow I_0^*$ where $0 \rightarrow N \rightarrow I_0 \rightarrow I_1$ is a minimal injective copresentation of N . We obtain using exactness of D ,

$$0 \rightarrow DN^* \rightarrow DI_0^* \rightarrow DI_1^* \rightarrow D\text{Tr}N \rightarrow 0,$$

i.e. $0 \rightarrow \nu N \rightarrow \nu I_0 \rightarrow \nu I_1 \rightarrow \mathrm{DTr} N \rightarrow 0$. Since the I_i^* are injective and hence projective left comodules, the νI_i^* are injective, and we have a minimal injective resolution of νN . Thus $\Omega^{-2}\nu(N) = \mathrm{DTr} N$. This proves (a).

To prove (b) we have by definition of the cosyzygy,

$$0 \rightarrow N \rightarrow I_0 \rightarrow I_1 \rightarrow \Omega^{-2}(N) \rightarrow 0$$

and by exactness of $*$,

$$0 \rightarrow (\Omega^{-2}N)^* \rightarrow I_1^* \rightarrow I_0^* \rightarrow N^* \rightarrow 0$$

whence $(\Omega^{-2}N)^* \cong \mathrm{Tr} N$; applying D yields the result.

Assume next that C is symmetric. Let R denote the dual algebra DC . By definition, C embeds in R as a R, R -bimodule and by [CDN, Theorem 5.3], we have

$$\mathrm{Hom}_{\mathbb{k}}(N, \mathbb{k}) \cong \mathrm{Hom}_R(N, R),$$

an isomorphism of right R -modules, which is natural in N . This can be seen by noting that C embeds in R as a left R -module as $\mathrm{Rat}({}_R R)$, the largest right rational left R -submodule of R . Then $\mathrm{Hom}_{\mathbb{k}}(N, \mathbb{k}) \cong \mathrm{Hom}_R(N, C) \cong \mathrm{Hom}_R(N, \mathrm{Rat}(R)) \cong \mathrm{Hom}_R(N, R)$ (the first isomorphism sends f to $(id_N \otimes f)\rho_N$).

We need to show that the composition $\mathrm{D}(-)^*$ is naturally isomorphic to the identity functor. By the Lemma, $N^* = \oplus \mathrm{D} \mathrm{Hom}(P_i, N)$ where the P_i are projective indecomposable right subcomodules of C . Since N is of finite length, $\mathrm{Hom}(P_i, N)$ is nonzero for only finitely many i . It follows (by taking coefficient space of $\oplus P_i$) that $\mathrm{D} \mathrm{Hom}_{\mathrm{DC}}(C, N) \cong \mathrm{D} \mathrm{Hom}_{\mathrm{DC}}(F, N)$ for some finite-dimensional subcoalgebra $F \subset C$. Thus may assume C is finite dimensional and we have isomorphisms

$$\begin{aligned} N^* &= \mathrm{D} \mathrm{Hom}_R(C, N) \\ &\cong \mathrm{D} \mathrm{Hom}_R(\mathrm{DN}, R) \\ &\cong \mathrm{D} \mathrm{Hom}_{\mathbb{k}}(\mathrm{DN}, \mathbb{k}) \\ &= \mathrm{D}^3(N) \\ &\cong \mathrm{DN}, \end{aligned}$$

which are natural in N . Therefore ν is isomorphic to the identity functor on \mathcal{M}_f^C . This completes the proof of (c). ■

Thus, by results of [CKQ], if N is an indecomposable non-injective finite-dimensional comodule over a symmetric coalgebra, then the almost split sequence starting at N ends at $\Omega^{-2}(N)$. Dually, if M is an indecomposable non-injective finite-dimensional comodule over a symmetric coalgebra, then the almost split sequence ending at M starts at $\Omega^2(N)$.

2 Examples

2.1 \mathbb{A}_∞^∞

Let Q be the quiver of type \mathbb{A}_∞^∞ with vertices indexed by \mathbb{Z} and arrows $a_i : i \rightarrow i+1$ for all $i \in \mathbb{Z}$. Fix $n > 0$ and consider the subcoalgebra C of the path coalgebra $\mathbb{k}Q$ spanned by paths of length at most n . Thus C is the degree n term of the coradical filtration of $\mathbb{k}Q$.

The coalgebra C is a serial coalgebra [CG2] and each of its finite-dimensional indecomposable representations is isomorphic to the right coideal generated by a path of some fixed length $\ell \leq n$. In detail, fix integers $i \leq j$ and let $V_{i,j}$ be the right coideal spanned by subpaths $a_j \cdots a_t$, $i \leq t \leq j$, all of which end at j (paths are written from right to left). Then $V_{i,j}$ is the right coideal generated by the path $a_j \cdots a_i$ of length $j - i \leq n$ where $V_{i,i}$ is regarded as the simple comodule at the vertex i . As a representation of Q , $V_{i,j}$ has a one-dimensional vector space at the vertices between i and j with identity maps assigned to the arrows. In particular $S(i) = V_{i,i}$ is the simple right comodule at the vertex i . Dually, let $U_{i,j}$ be the left coideal spanned by the subpaths $a_t \cdots a_i$ of $a_j \cdots a_i$. The $U_{i,j}$ are the indecomposable left comodules, which correspond to indecomposable representations of Q^{op} , and $U_{i,i}$ is the simple left comodule at the vertex i . Clearly, $U_{i,j} = DV_{i,j}$. It is easy to see that $I_i = V_{i-n,i}$ is both the injective envelope of $S(i)$ and the projective cover of $S(i-n)$. Evidently, C is a self-projective coalgebra. But C is not a symmetric coalgebra because the Nakayama functor is not isomorphic to the identity as we show presently.

Proposition 2.1.1 (a) $\Omega^{-2}(V_{i,j}) = V_{i-n-1,j-n-1}$

(b) $\nu(V_{i,j}) = V_{i+n,j+n}$

(c) $DTr(V_{i,j}) = V_{i-1,j-1}$

(d) $\Omega^{-2}(U_{i,j}) = V_{i+n+1,j+n+1}$

(e) $\nu(U_{i,j}) = V_{i-n,j-n}$

(f) $DTr(U_{i,j}) = V_{i+1,j+1}$

Proof. We have the injective copresentation

$$0 \rightarrow V_{i,j} \rightarrow V_{j-n,j} \rightarrow V_{i-1-n,i-1} \rightarrow \Omega^{-2}(V_{i,j}) \rightarrow 0$$

which yields $\Omega^{-2}(V_{i,j}) = V_{i-n-1,j-n-1}$ by inspection. This proves (a). Dualizing, we have

$$0 \rightarrow \text{Tr } V_{i,j} \rightarrow V_{i-1-n,i-1}^* \rightarrow V_{j-n,j}^* \rightarrow V_{i,j}^* \rightarrow 0$$

which we rewrite as

$$0 \rightarrow \text{Tr } V_{i,j} \rightarrow U_{i-1,i-1+n} \rightarrow U_{j,j+n} \rightarrow V_{i,j}^* \rightarrow 0$$

By inspection this yields $\text{Tr } V_{i,j} = U_{i-1,j-1}$ and $V_{i,j}^* = U_{i+n,j+n}$. Assertions (b) and (c) follow immediately. The dual assertions (d-f) are similar. ■

The almost split sequences are just as for Nakayama algebras [ARS] as follows:

$$0 \rightarrow V_{i,j} \rightarrow V_{i-1,j} \oplus V_{i,j-1} \rightarrow V_{i-1,i-1} \rightarrow 0$$

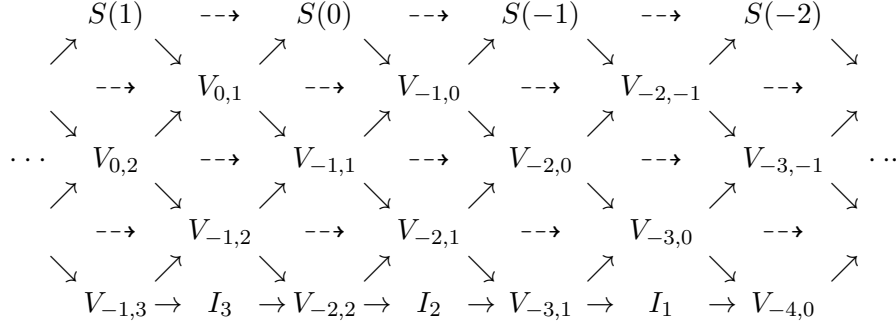
if $n > i - j > 0$. The boundary sequences are

$$0 \rightarrow S(i) \rightarrow V_{i-1,i} \rightarrow S(i-1) \rightarrow 0$$

and

$$0 \rightarrow V_{i-n+1,i-1} \rightarrow I_i \oplus V_{i-n+1,i-1} \rightarrow V_{i-n,i-1} \rightarrow 0.$$

The AR quiver for $n = 4$ is shown below. The stable AR quiver is of type $\mathbb{Z}\mathbb{A}_n$. The translation is denoted by \dashrightarrow .



2.2 Quantum $SL(2)$ at a root of 1

Assume the base field \mathbb{k} is of characteristic zero. Henceforth, let $C = \mathbb{k}_\zeta[SL(2)]$ be the q -analog of the coordinate Hopf algebra of $SL(2)$, where q is specialized to a root of unity ζ of odd order ℓ .

The coalgebra C has the following presentation. The algebra generators are a, b, c, d , with relations

$$\begin{aligned} ba &= \zeta ab \\ db &= \zeta db \\ ca &= \zeta ac \\ bc &= cb \\ ad - da &= (\zeta - \zeta^{-1})bc \\ ad - \zeta^{-1}bc &= 1 \end{aligned}$$

and with Hopf algebra structure further specified by

$$\begin{aligned} \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -\zeta b \\ -\zeta^{-1}c & a \end{pmatrix} \end{aligned}$$

By general coalgebra theory (see [Ch]), C is the direct sum of blocks, each determined by an equivalence class (under linkage) of simple comodules.

For quantum $SL(2)$, each simple is labelled by a nonnegative integer, called a *highest weight*. In [Ch2] we see that there are precisely $\ell - 1$ nontrivial blocks $C_0, \dots, C_{\ell-2}$ where C_r contains the simple comodule of highest weight r , $0 \leq r < \ell - 2$. The other simples in C_r are obtained via ℓ -reflections. There are an infinite set of trivial blocks, whose simple modules are projective and injective and whose quivers therefore consist of a single vertex. Thus in some sense the representation theory of C is reduced to the nontrivial blocks, which turn out to be Morita-Takeuchi equivalent to each other. A nontrivial block of the basic coalgebra is described as a subcoalgebra of the path coalgebra of its quiver in the next result.

Theorem 2.2.1 ([Ch2]) *The basic coalgebra B of C_r is the subcoalgebra of path coalgebra of the quiver*

$$\begin{array}{ccccccc} & b_0 & & b_1 & & b_2 & \\ 0 & \xleftrightarrow{\quad} & 1 & \xleftrightarrow{\quad} & 2 & \xleftrightarrow{\quad} & \dots \\ & a_0 & & a_1 & & a_2 & \end{array}$$

spanned by the group-likes g_i corresponding to vertices and arrows a_i, b_i together with coradical degree two elements

$$\begin{aligned} d_0 &:= b_0 a_0 \\ d_{i+1} &:= a_i b_i + b_{i+1} a_{i+1}, \quad i \geq 0. \end{aligned}$$

Proposition 2.2.2 *C is a symmetric coalgebra.*

Proof. By a result of Radford, there is a distinguished group-like element in C (see [DNR, p. 197]), which is trivial if and only if C is unimodular. There is a unique one-dimensional comodule, namely the trivial comodule, so C is unimodular. This implies that the distinguished group-like element in C is just the identity element. The square of the antipode of U_ζ is inner by the element $K^2 \in U_\zeta \subset DC$, and it follows by duality that the square of the antipode of C is given by $S^2(c) = K^{-2} \rightharpoonup c \leftarrow K^2$ (a fact that can be checked directly). Thus C is unimodular and is inner by an element of DC , and the conclusion now follows from [CDN]. ■

Remark 2.2.3 *The conclusion can be similarly shown to hold more generally for quantized coordinate algebras, using the fact that S^2 is inner in the generic quantized enveloping algebra $U_\zeta(\mathfrak{g})$ where \mathfrak{g} is complex semisimple Lie algebra, see e.g., [Ja] 4.4.*

The fact that each nontrivial block B is a symmetric coalgebra follows from

Lemma 2.2.4 *The basic coalgebra of a symmetric coalgebra C is symmetric.*

Proof. By definition [CDN], there is a DC -bimodule embedding $C \rightarrow DC$. By [CG] the basic coalgebra B of C is of the form eCe where $e \in DC$ (left and right "hit" actions). One can easily check that $DB \cong e(DC)e$, so that the coalgebra eCe embeds in $D(eCe)$ as a DB, DB -bimodule by restriction. ■

Remark 2.2.5 A symmetrizing form (see [CDN]) $\phi : B \rightarrow \mathbb{k}$ can be defined by $\phi(d_i) = 1$ and $\phi(a_i) = \phi(b_i) = \phi(g_i) = 0$ for all i . This gives another proof that each nontrivial block B is a symmetric coalgebra. This form is an associative, symmetric, nondegenerate, DB-balanced form as required.

The indecomposable injective right B -comodules are the coideals

$$\begin{aligned} I_n &= \mathbb{k}g_n + \mathbb{k}a_{n-1} + \mathbb{k}b_n + \mathbb{k}d_n \\ I_0 &= \mathbb{k}g_0 + \mathbb{k}b_0 + \mathbb{k}d_0 \end{aligned}$$

($n \geq 1$) and they are also projective comodules. Clearly $\text{rad}I_n = \mathbb{k}g_n + \mathbb{k}a_{n-1} + \mathbb{k}b_n$ and $\text{rad}I_0 = \mathbb{k}g_0 + \mathbb{k}b_0$. Let $S(n)$ the simple right comodule corresponding to the vertex g_n , for all $n \in \mathbb{N}$. The non-injective finite-dimensional comodules are described in [Ch2] as *string comodules* and can be computed using cosyzygies and syzygies:

Proposition 2.2.6 ([Ch2]) *Every finite-dimensional non-injective indecomposable B -comodule is in the Ω^\pm -orbit of some simple comodule.*

The almost split sequences and AR quiver for the category of finite-dimensional B -comodules are described starting with the sequences having an injective-projective comodule I_n in the middle term. These are precisely the sequences (see e.g. [ARS, p. 169])

$$0 \rightarrow \text{rad}(I_n) \rightarrow \frac{\text{rad}(I_n)}{\text{soc}(I_n)} \oplus I_n \rightarrow \frac{I_n}{\text{soc}I_n} \rightarrow 0 \quad (1)$$

with $n \in \mathbb{N}$. These sequences can be rewritten as

$$0 \rightarrow \Omega(S(n)) \rightarrow S(n-1) \oplus S(n+1) \oplus I_n \rightarrow \Omega^{-1}(S(n)) \rightarrow 0 \quad (2)$$

where $S(-1)$ is declared to be 0.

Theorem 2.2.7 ([Ch2]) *Applying Ω^i , $i \in \mathbb{Z}$, to the sequences (2) yields all almost split sequences for \mathcal{M}_f^B .*

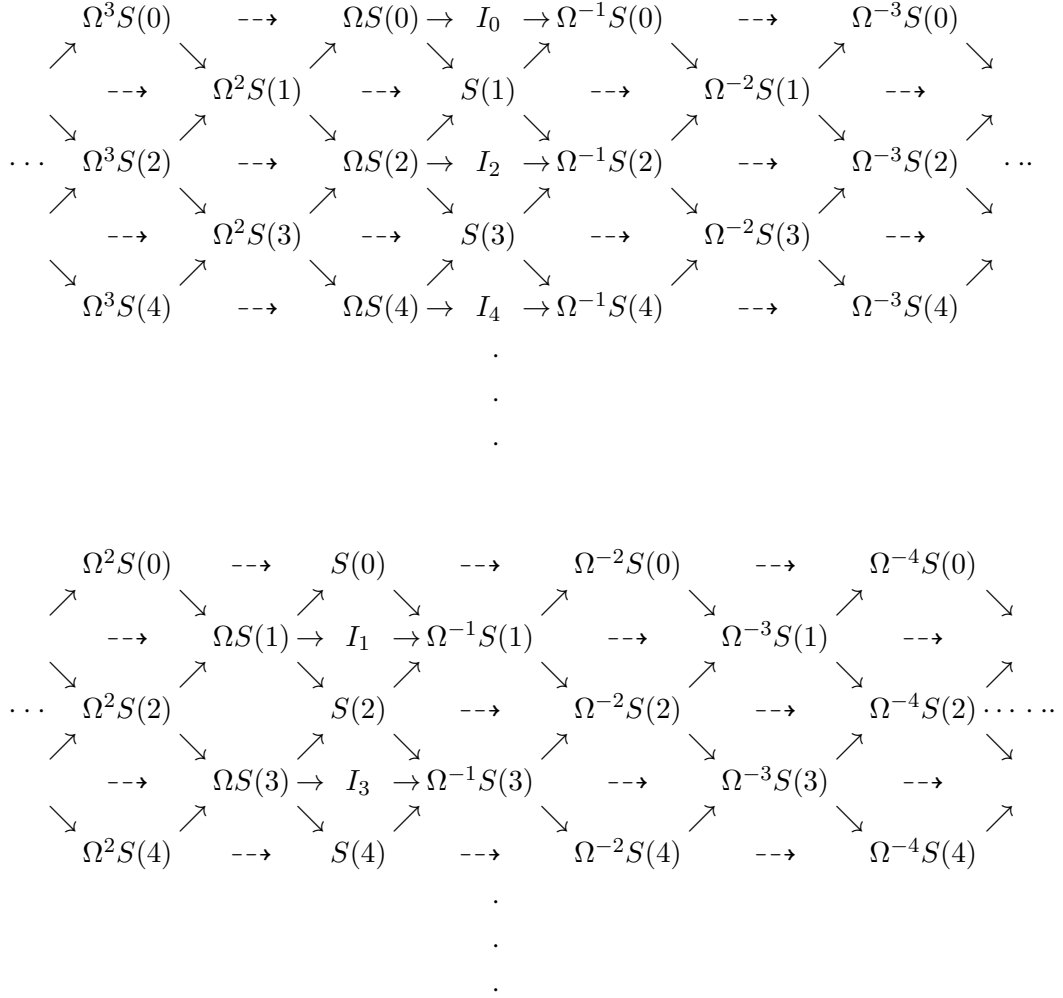
The sequences (2) are precisely the almost split sequences with an injective-projective in the middle term; the other sequences

$$0 \rightarrow \Omega^{i+1}(S(n)) \rightarrow \Omega^i S(n-1) \oplus \Omega^i S(n+1) \rightarrow \Omega^{i-1}(S(n)) \rightarrow 0$$

we obtain when we apply Ω^i , for a nonzero integer i , are the almost split sequences without an injective-projective summand in the middle term.

We define the Auslander-Reiten (AR) quiver of a coalgebra to be the quiver whose vertices are isomorphism classes of indecomposable comodules and whose (here multiplicity-free) arrows are defined by the existence of an irreducible map between indecomposables.

We finally display the AR quiver for \mathcal{M}_f^B . The stable AR quiver (with all injectives deleted) consists of two components of type $\mathbb{Z}\mathbb{A}_\infty$ which are transposed by Ω .



References

- [ARS] M. Auslander, I. Reiten, S. Smalø, *Representation Theory of Artin algebras*. Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, 1997.
- [ASS] I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory*. London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006.
- [Ch] W. Chin, A brief introduction to coalgebra representation theory, in: Proceedings from an International Conference Held at DePaul University, J. Bergen, S. Catoiu, W. Chin, eds. Marcel Dekker (2003).

- [Ch2] W. Chin, Special biserial coalgebras and representations of quantum $SL(2)$, preprint.
- [CDN] S. Dăscălescu, F. Castaño Iglesias and C. Năstăsescu, Symmetric coalgebras, *J. Alg.* 279 (2004), 326-344.
- [CG] J. Cuadra Diaz, J. Gómez Torrecillas, Idempotents and Morita-Takeuchi theory. *Comm. Algebra* 30 no. 5, (2002), 2405–2426.
- [CG2] J. Cuadra Diaz and J. Gómez Torrecillas, Serial Coalgebras, *Journal of Pure and Applied Algebra*, 189, (2004) 89-107
- [CKQ] W. Chin, M. Kleiner and D. Quinn, Almost split sequences for comodules, *J. Alg.* 249 (2002) no. 1, 1-19.
- [CKQ2] W. Chin, M. Kleiner and D. Quinn, Local theory of almost split sequences for comodules, *Ann. Univ. Ferrara - Sez. VII - Sc. Mat.* Vol. LI, 183-196 (2005).
- [CS] W. Chin and D. Simson, Coxeter transformation and inverses of Cartan matrices for coalgebras, *J. Algebra*, 324 no. 9, (2010), 2223-2248
- [DNR] S. Dăscălescu, C. Năstăsescu, Ş. Raianu, Hopf Algebras: an introduction. Vol. 235. *Lecture Notes in Pure and Applied Math.* Vol.235, Marcel Dekker, New York, 2001.
- [Iov] M. Iovanov, Abstract Integrals for Hopf Algebras, preprint 2010.
- [Ja] J. C. Jantzen, *Lectures on Quantum Groups*, AMS Graduate Studies in Mathematics, vol. 6 (1996).